Stimulating Mathematics-in-Industry

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1 Introduction

It is a wonderful surprise and honour to be awarded the IMA Gold Medal. It is especially poignant for me because one of the people who most influenced my career was my supervisor Alan Tayler, who was the recipient (in 1982) of the first medal, jointly with another of my role models, James Lighthill. Moreover, the theme of my talk is an activity that would have been very different had it not been for the vision of these two people.

In 1965, James Lighthill was in part responsible for a report, sponsored by the Royal Society, in which it was noted how strongly UK applied mathematics was rooted in theoretical mechanics, and the report suggested that maybe it was time to broaden the range of applications. Alan Tayler, applied mathematician and rugby player, and his colleague Leslie Fox, numerical analyst and golf player, seized on this suggestion and proposed to the Science Research Council to hold a series of "Study Groups". These were quite unlike the then British Theoretical Mechanics Colloquia and their nucleus was the presentation, by a handful of industrial researchers, of open problems to a group of open-minded academic mathematical scientists. These academics were initially all from Oxford but the resulting constraints in expertise were soon recognised, especially in Nottingham, and help in all areas of mathematical expertise was soon forthcoming from around the country. What really made the Study Group concept work was that, after the presentations, the academics were left to gravitate to those problems they found to be of greatest interest. Their ideas were then reported back on the final day, as indicated in the representative programme in Figure 1.

Finally, and crucially for the industrial impact, an accessible technical report was sent to each industrial participant within a few weeks of the Study Group.

	Speaker	Lecture
Monday, 15th March		
9.30	Mr Morton	C.E.G.B. Problem (cooling towers)
11.15	Mr Reagan	I.C.I. Problem (reactors)
2.30	Dr Tough and Dr Leigh	B.I.S.R.A. Problem (rolling steel)
4.15	Mr Jenkins	Pilkingtons Problem (rolling glass)
Tuesday, 16th March		
9.30	Professor L. Fox	Ill conditioned numerical problems and instabilities of numerical methods
11.15	Discussion Groups	
2.30	Dr J.D. Murray	Mathematical Modelling
4.15	Discussion Groups	
Wednesday, 17th March		
9.30	Dr A.B. Tayler	The use of analytical methods before extensive computation
11.15	Discussion Groups	
2.30	Dr D.F. Mayers and Miss J. Taylor	Numerical methods involving free boundary problems
4.15	Discussion Groups	
Thursday, 18th March		
9.30	Dr J.R. Ockendon and Dr H. Ockendon	Wave propagation and partial differential equations
11.15	Free	
2.30	Dr Donnelly and Dr Grant	Numerical methods for partial differential equations
4.15	Free	
Friday, 19th March		
9.15	Discussion of B.I.S.R.A. Problem	
10.00	Discussion of I.C.I. Problem	
11.15	Discussion of C.E.G.B. Problem	
12.00	Discussion of Pilkington Problem	
2.30	Further discussion time and review of Study Group organisation	

Figure 1

I must add that the intensity of a Study Group can only be appreciated by the participants, who are frequently moved to rivalry, and even abuse, but, more frequently, to the utterance of "Colemanballs", made famous in [10].

More or less everything I now have to say concerning stimulation has resulted from the Study Group concept.

2 The Pantograph equation

My eyes were first opened to the academic stimuli that Study Groups could offer when, at the second meeting, the problem illustrated in Figure 2 arrived from the then British Railways.

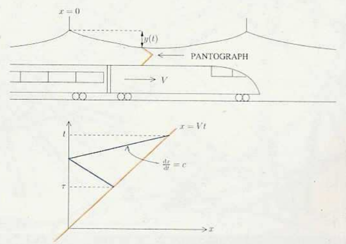
Figure 2: **Problem**: To model and redesign the overhead electricity collection system for a train pantograph to ensure contact is maintained throughout.

The equation

$$\frac{dy}{dt}(t) = ay(\lambda t) + by(t), \qquad t > 0, \qquad (1)$$

which had rarely been written down before (see the references in [5]), is called the pantograph equation, and it now has a burgeoning literature; its derivation is sketched in Figure 3.

It offers wonderful analytical and numerical possibilities and, as an asymptotic challenge, Alan and I focussed on the case



$y(t)$ depends on $y(\tau)$ where

$$t - \tau = \frac{V}{c}(t + \tau).$$

Hence $dy(t)/dt$ depends on $y(\lambda t)$ where

$$\lambda = \frac{c - V}{c + V} < 1.$$

$$\frac{dy(t)}{dt} = ay(\lambda t) + by(t)$$

is the Pantograph equation (Ambartsumian, 1944).

Figure 3

$\lambda = 1 - \varepsilon$, where ε is small. The method of multiple scales can then be used to show that the solution of

$$\frac{dy}{dt}(t) = 0.95\,y(0.99t) - y(t), \qquad y(0) = 1 \qquad (2)$$

is as in Figure 4. The sudden appearance of unbounded oscillations when $t = 0(10^2)$ enabled us to pose Leslie and his colleague David Mayers a severe numerical challenge. It also gave the first indication of the wide variety of parameter regimes catalogued in [8], which subsequently stimulated the beautiful analysis of [11] and led to the "phase planes" of $\Re y$, $\Im y$ in Figure 5. The pantograph equation remains a problem of current interest, as I will explain later.

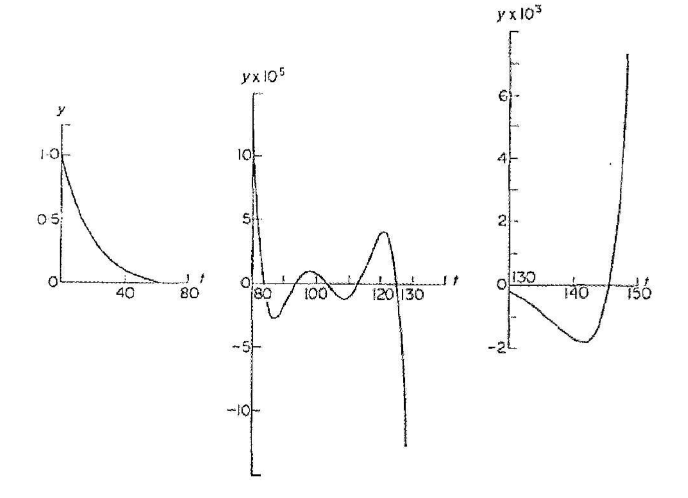


Figure 4

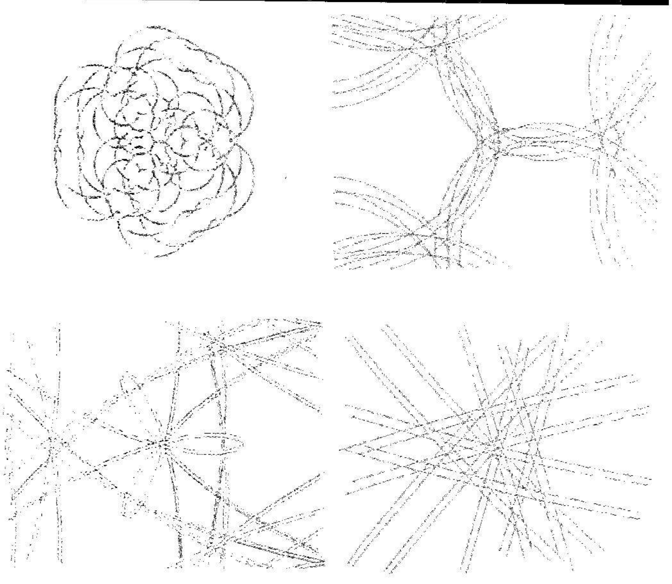


Figure 5: Solutions of $y''(t) + y(t) = 5iy'(t/4)$ in the 'windows' 50×10^{-j}, $j = 0,1,3,4..$

3 Some definitions

If the concept of mathematics-in-industry is to gain credibility in the eyes of academics, industry or government, it is crucial to define its distinctiveness. I like the Brussels definition of industry as "any activity of economic or social value" and, almost by definition, industrial research is interdisciplinary. But what distinguishes mathematics-in-industry within applied mathematics?

Applied mathematics encompasses any mathematics that is relevant to the real world; it is often driven by theoretical developments, but more often by curiosity about real world phenomena, from what happens when you toast your bread in the morning to what happens when you brush your teeth at bedtime. Mathematics-in-Industry on the other hand is nothing if it is not industry-driven. This implies that it is stimulated by a non-academic who typically wishes to benefit from a quantitative study of a process, and the value of this study is not usually expressible in terms of any academic norm. It is interesting to contrast mathematics-in-industry with, say, mathematical biology; there, it is usually another academic who has the greatest chance of benefitting from the study, which can therefore be carried out on academic timescales. Mathematics-in-Industry has a much greater urgency about it and, with this, comes possible accusations of shallowness, which is another topic to which I will be returning.

4 The mathematical dividend

It soon turned out that the pantograph problem was the tip of an iceberg when it came to stimulating new mathematics. For me, the most important substantive new area to be opened up was the class of differential equations problems called free boundary problems. It was a direct result of the appearance of models for welding, glass furnaces, ingot solidification and iced lollipop manufacture at Study Groups that spawned the first ever conference on that topic, organised by the IMA [13]. It rapidly became apparent that these problems cried out for a mathematical unification as dramatic as the ideas of ellipticity and hyperbolicity for partial differential equations, and there are now tens of thousands of papers on the subject. My two paradigms are the following.

An early problem from the long-vanished British Motor Corporation concerned resistance welding. At first sight this seemed a straightforward generalisation of the type of one-dimensional "Stefan problem" described in chapter XI of [6]; its only novelty was that heat was supplied volumetrically via Joule heating rather than by conduction from the boundary of the sample. Since it seemed to pose a relatively routine numerical problem for the free boundary between the solid and the weld nugget, Alan and I asked our student, David Atthey, to proceed by using a crude finite difference scheme in which the free boundary passed through nodal points. He did this over a period of months, but always ended up by predicting that the solid metal became superheated, i.e. that its temperature exceeded the melting

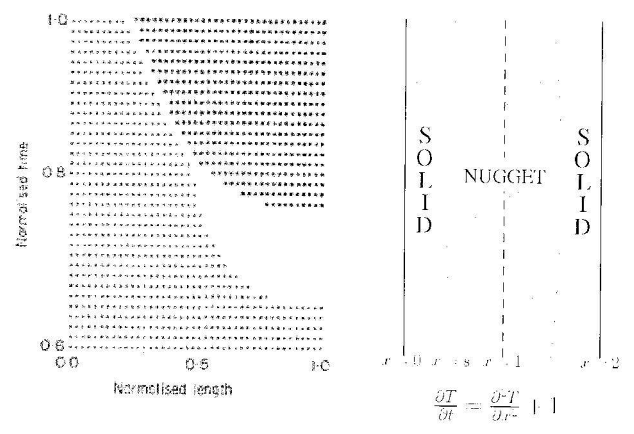


Figure 6: The enthalpy method computations for the x, t plane showing the solid, mushy and liquid regions, and the naive free boundary problem for $x = s(t)$.

temperature. It was only when Robin Hodgkins suggested that David use the numerical "enthalpy method" that the physically acceptable solution emerged, as illustrated in Figure 6.

The fact that the "enthalpy discretisation" converges to the weak solution of the free boundary problem, which is quite different to the classical, superheated, solution, provided a powerful boost for the whole theory of free boundary problems [4].

Even more far-reaching is the "Hele-Shaw" problem for the free boundary of the liquid in a partially filled Hele-Shaw cell, which consists of two parallel plates, separated by a narrow gap. This problem has largely forgotten both its engineering genesis and the stimulus it received by virtue of its being a special case of the two-dimensional Stefan problem. Indeed, nowadays physicists, and others who prefer to think of the complement of the fluid region as that region of the Argand diagram occupied by the eigenvalues of large random normal[1] matrices, refer to it as the problem of "Laplacian growth" [12]. But, when the fluid region in the cell is contracting, it remains the prototype ill-posed model of a configuration that is trivial to realise practically (see Figures 7 and 8, taken from [9]).

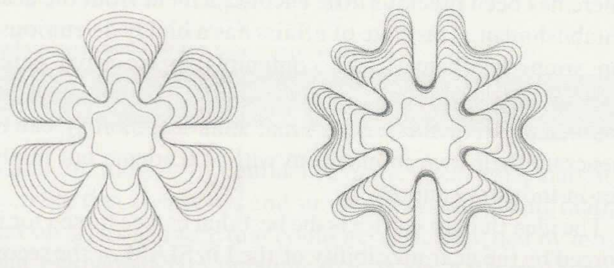


Figure 7: Analytical solutions for radial fingering: the fluid region is external to this evolving free boundary.

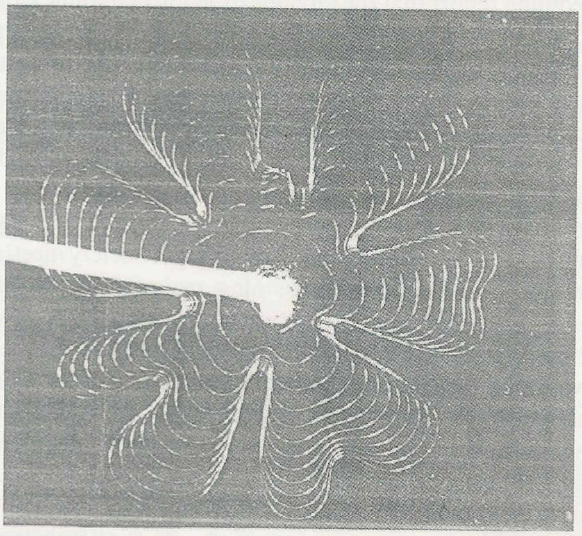


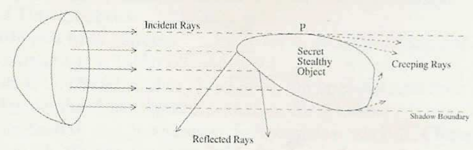
Figure 8: Air injected into a Hele-Shaw cell.

Apart from free boundary problems, other major topics that have received strong but less striking stimuli from industry include (i) generalised parabolic partial differential equations for thin film flows arising, say, in glass, coating, lubrication and nowadays microfluidics, and (ii) stochastic differential equations, which were much more esoteric before the days of mathematical finance. In Oxford, a boost for the latter came not from the City but from the leisure industry, when a company called IG Index asked for help in understanding the valuation of options. Along with these developments in continuum mathematics have come new challenges for discrete mathematics especially in the theories of graphs and networks and image and data processing. Although I am well-known for not being a discrete mathematician, I was quite fascinated by the problem of making office blocks secure for wireless communication; in other words how many colours do you need to paint cuboidal offices in a block so that no wall has the same colour front and back.

When we played around with this under pressure at a Study Group, the answer seemed to be about 6 but in fact, for n offices, it can even be proved [14] that you need at least of order $\log(\log n) / (\log(\log(\log n)))$ colours.[2] It is interesting that BAe Systems, whose problem this was, had, a decade earlier, challenged the mathematics community to find, to high accuracy, the radar field scattered by a stealthy aircraft (Figure 9).

EXPONENTIAL ASYMPTOTICS

Real World Problem: Radar Scattering



Need to predict back scatter from creeping rays: $\Rightarrow$

creeping field is $O(e^{-k^{1/3}})$, $k = \omega L/c$

But in some cases terms of $O(e^{-k})$ are important too.

Hence BOOST for COMPLEX RAY THEORY

BASIC IDEA: Use WKB with $\phi \sim Rd\, Ae^{iku}$ and allow x, y, u to be complex. Then derive "switching rules" for which fields u_i are acceptable. Switching can only occur at Stokes lines where $Rlu_i = Rlu_j$ but not all Stokes lines are "active".

Figure 9: Exponential Asymptotics
Real World Problem: Radar Scattering.

This gave a great stimulus to the little-studied field of complex ray theory which is one of the cornerstones of the exciting but incredibly complicated subject of exponential asymptotics. Figure 10 shows the geometry of the Stokes lines that demarcate the different regions of existence of the waves, most of which are exponentially small in terms of wavenumber, near the point of grazing incidence on a circular cylinder [7].

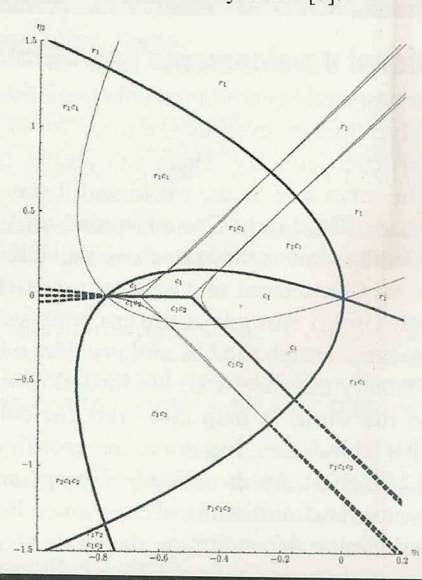


Figure 10: Stokes structure near the tangency point of a smooth cylindrical scatterer.

[1] A normal matrix is one that commutes with its transpose.

[2] This gave rise to the Colemanball "You can always take the ceilings to be vertical and the walls to be horizontal".

5 The industrial dividend

I would like to be able to say more about the benefits that at least some of these intellectual advances have brought to industry. There is no shortage of glowing comments, such as those quoted in glossy roadmaps and other publicity exercises [1]. But industry is naturally reluctant to discuss profits and the best I can do is pass on rumours of how a drawing instability revealed at a Study Group saved a now extinct man-made fibre manufacturer hundreds of thousands of pounds and how a long-forgotten steel manufacturer made even more by selling a blast-furnace code based on an algorithm devised at a Study Group. Even more speculative is the estimate of the value of mathematics-in-industry to the UK economy quoted in [3]. The example I like the best is the possibility of a patent for a small company in the milk industry concerning the automatic measurement of milk flow from an udder, as illustrated in Figure 11.

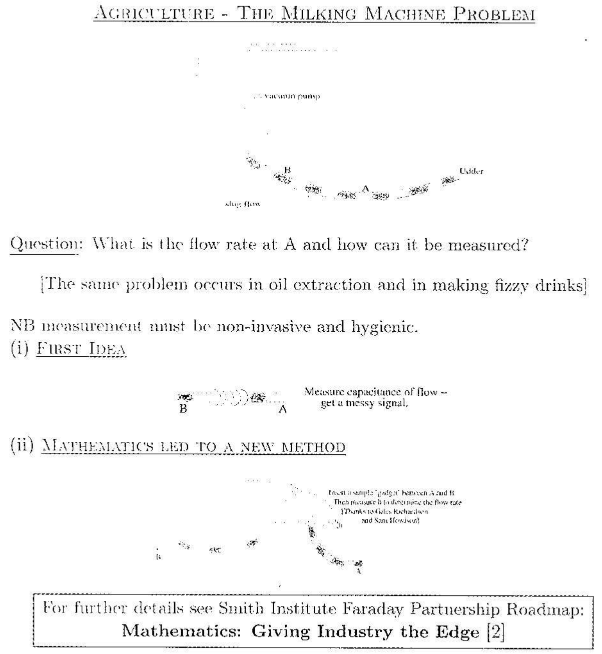


Figure 11: Agriculture - The Milking Machine Problem.

6 International developments and infrastructure

All this burgeoning activity could never be sustained by a mere handful of universities or even by the majority of universities within any particular country. There will always be untapped expertise on the other side of the world and it was a great step forward when, in 1984, Study Groups spread to Australia and USA. They also became national across the UK from 1989, shortly to become Continental, and we have just reached the 60th European Study Group. Nowadays, fifteen countries have annual Study Groups, even though the UK still provides a hard core of twenty or so faculty members who are invited to make regular forays around the world to help kick start the collaborations. Along with this stimulation, has come the growth of clinics in Claremont (California), Medical Study Groups and In-house Study Groups at several multinational companies. But it is not all workshops and clinics: depending on the levels of support and the enthusiasm of industry, many kinds of intermediate institutes have sprung up such as the Smith Institute[3] (managers of the Faraday Partnership, and now the Knowledge Transfer Network) in the UK, MITACS in Canada, and the Fraunhöfer Institute and Matheon in Germany. They all have in common strong leadership and vision and dedicated support staff, such as "technology translators" for the KTN, who can really help to take the administrative burden off academic shoulders. However, their different modi operandi result from how their funding is split between public and industrial sources.

7 The other side of the coin

Despite all the success and mutual benefits that have attended the growth of mathematics-in-industry, it always has and always will suffer from a daunting intellectual handicap. My first inkling of this came in 1970 when Keith Stewartson, an icon of the UK applied mathematics community, asked me why I should indulge in such an activity when there were enough problems in fluid mechanics to keep us all going for decades (and this was certainly an accurate prediction). Later I had to ask myself why Alan Tayler, IMA Gold Medallist and holder of the CBE had received no recognition from within his university for his towering achievements. Indeed for the "hard core" I mentioned above there has been precious little encouragement from the academic establishment. This state of affairs has a highly deleterious effect on young mathematicians contemplating research careers in mathematics-in-industry; why should it be that other applied mathematical themes, such as mathematical biology, can readily generate their own momentum within academe but mathematics-in-industry cannot?

The idea that lip service is the best that can be hoped for is reinforced by the near invisibility of the I in SIAM or the second I in ICIAM and, indeed there is no doubt that the lip service phenomenon is truly international. This point will be made in a forthcoming report that the Organisation for Economic Cooperation and Development commissioned to advise their member governments of the importants of mathematics-in-industry. Delegates from 15 countries were unanimous in identifying the academic disincentives for mathematics-in-industry, which range from difficulties in publishing[4] to a lack of dedicated academic positions.

My diagnosis of the reason for this attitude towards interdisciplinarity results from a recent Royal Society meeting on the subject. There the overwhelming majority of leaders of research groups across academic science expressed the view that they could "do interdisciplinarity" within the existing framework of resources and departmental structures. Such an attitude is well founded in areas like materials science and earth science, which are interdisciplinary by nature, but can the same be said for mathematics? Maybe the attitude is related the fact that round the world no more than 20% of any applied mathematics community ever seem to have a genuine appetite for the interdisciplinarity inherent in mathematics-in-industry. Its consequence is that the number of heads of department who will ever encourage modelling, problem solving and simulation, at the possible expense of prestige measured by traditional norms, is even tinier. Nonetheless, there is a significant minority of talented mathematicians who thrive under the stimulus of mathematics-in-industry; why should there be no career paths for young researchers in this category and why, at a time when connectivity is a buzzword, should funding opportunities for mathematics-in-industry be so hard to come by?[5]

[3] See the accompanying interview.

[4] The Journal in [14] is a very first step from Canada aimed at remedying this.

[5] This is discussed further in the accompanying article by Alasdair Rose.

8 Hopes for the future

I am quietly confident that mathematics-in-industry will continue to thrive for two reasons that transcend the difficulties mentioned above, even though at least some of these difficulties may eventually prove illusory.

$$x = \text{cash at } t = 0;$$
$$T = \text{time to bankruptcy};$$
$$V = \text{spending rate};$$

Bets are made at t_i, $i = 1, 2, \ldots$ with $\tau_i = t_i - t_{i-1}$.

$$P\{\tau > t\} = e^{-\mu t}$$

Probability of bankruptcy is $f(x)$ where

$$\begin{aligned} f(x) &= P\{T < \infty\} \\ &= P\{t_1 > x/V\} + P\{t_1 < x/V\} \\ &= e^{-\mu x/V} + \int_0^{x/V} \mu e^{-\mu s} P\{T < \infty | t_1 = s\} \mathrm{d}s \\ &= e^{-\mu x/V} + \frac{1}{2} \int_0^{x/V} \mu e^{-\mu s} \Big(f(\lambda(x - Vs)) + f(\lambda^{-1}(x - Vs)) \Big) \mathrm{d}s, \end{aligned}$$

where λ = gain from bet, λ^{-1} = loss from bet.
Differentiation $\Rightarrow$

$$\frac{V}{\mu} f'(x) + f(x) = \frac{1}{2} \Big(f(\lambda x) + f(\lambda^{-1} x) \Big)$$

- a simple generalisation of the pantograph equation.

Figure 12 The Pantograph Equation Revisited - "Double or Half" Gambling Game.

First and foremost is my confidence in the intellectual viability of an activity that always has and surely always will spring mathematical surprises at a rate that could never be matched by most academic mathematicians pursuing their trade in the traditional way. Let me briefly return to the pantograph problem and please try to imagine my astonishment when I found my name on a draft [5] which was basically centred on Markov processes. As briefly outlined in Figure 12, the very asymptotic analysis that emerged from our studies in 1971 could be used to make a plausible conjecture concerning the ultimate bankruptcy of a gambler who, at random times, finds a (rare) bookmaker who will give him twice or one half his stake on the toss of a fair coin. I was delighted at this coming together of two such disparate branches of mathematics.

Secondly, and maybe more important practically, is the knock-on effect of the spread of mathematics-in- industry into countries with vast reserves of mathematical talent, especially India and China. As the popularity of hard mathematics (and mathematics-in-industry is hard) declines in many industrialised countries, mathematics-in-industry offers the UK, with its track record in the subject, a fantastic opportunity to attract talented young researchers who can so readily appreciate the key role that mathematics must play in this century. It is no good having brilliant problems without brilliant people to solve them.❑

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